Marcum Q-function as an analytical solution for misaligned Gaussian beams

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Abstract. A point-to-point optical link is usually based on a laser beam, where the most common propagation model is the Gaussian beam type. Due to its directional characteristics, minor alignment errors can cause considerable attenuation. The calculation of the misalignment losses involves solving an integral of a Gaussian beam translated over a receiver's effective surface, leading to a more accurate link budget. Our work proposes a closed-form expression for calculating the geometric and misalignment attenuation, caused by translational movements in circular Gaussian beams. © *2021 Society of Photo-Optical Instrumentation Engineers (SPIE)* [DOI: 10.1117/1.OE.60.5.056105]

Keywords: Gaussian beams; misalignment attenuation; optical wireless communications; closed form.

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1 Introduction

Optical communications are receiving special attention for being able to provide full support for the implementation of emerging telecommunication technologies. Nowadays, with the emergence of 5G networks and the expansion of the Internet of Things, the need for high-bandwidth increases. In parallel with fiber-optic communications, optical wireless communication (OWC) has gained significant importance due to their unique advantages: high bandwidth, license-free spectrum, high data rate, fast and easy deployment, low power requirements, as well as being able to operate on very light platforms.[1]

Most laser-based optical systems produce beams, whose irradiance profile can be modeled as Gaussian, such as in He-Ne lasers.[2] In optics, the Gaussian distribution is one of the most important field solutions for the Helmholtz paraxial equation, being considered the fundamental mode of propagation (TEM_{00}).[3] However, problems involving the detection of misaligned (off-axis) Gaussian beams require solving a surface integral, which does not have a closed-form solution. This problem is quite common when dealing with OWC, such as in free-space optical (FSO) communications[4] or chip-to-chip communications,[5] but it can also be applied in several other areas of optics, whenever the fundamental mode is transmitted in free space, such as in optical coupling or fiber splices.

The attenuation caused by a misaligned Gaussian beam is calculated through the integration of the beam distribution over the receiver's effective area. So far, this type of calculation has depended upon numerical solutions[6] or more complex expressions.[7–9] The most classical solution for this double integral involves the analytical solution of the internal integral, followed by a numerical solution to the integral of the resulting function.[7,8] Another work presented a closed-form solution involving infinite power series, which allows a faster implementation in some cases.[9]

This paper aims to present an analytical solution to attenuation caused by misalignment of a Gaussian beam, also considering the geometric attenuation, typical of OWC links. This approach can enable faster and more efficient calculations, especially for mobile optical systems.

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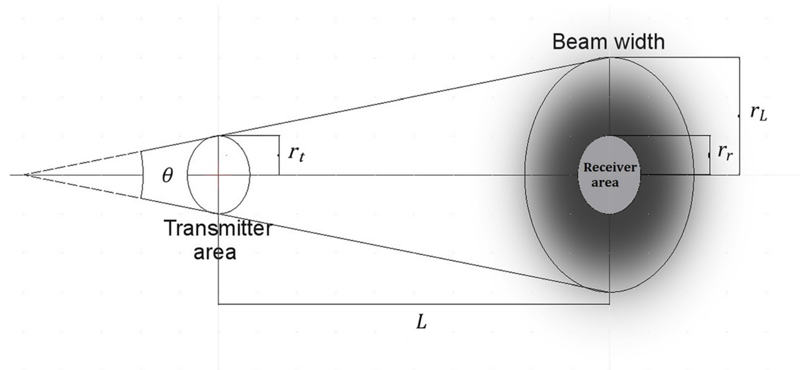

Fig. 1 Representation of a Gaussian beam propagation and its geometric attenuation.

2 Misaligned Gaussian Beam

Due to diffraction, even if a beam was perfectly collimated, its propagation through the atmosphere would not occur through a perfect cylindrical path. In reality, laser propagation is usually modeled using a very thin cone, whose opening angle is called the divergence angle. Thus, a portion of energy can be spread out and end up falling beyond the reception area as shown in Fig. 1. This type of attenuation, inherent in OWC links, is known as geometric attenuation. In a 1-km-long optical link, for example, a divergence angle of just 1 mrad will spread the energy of the signal along a circumference of 1 m in diameter.[10]

As OWC systems are normally employed with extremely directional beams, pointing errors can cause major attenuation. Considering long-range terrestrial links, such as FSO, even if the transceivers are over fixed structures, such as buildings, the inherent movement of the base cannot be disregarded. Buildings are subject to thermal expansions, winds, and vibration. Depending on the type of link, these small movements can be sufficient to displace the transceivers and interrupt communication.[4] In addition to the vibrational misalignment, atmospheric turbulence can generate the effect known as beam wander, which is the random wander of the beam's centroid over the reception plane.[11] Figure 2 shows both situations, when the receiver is perfectly aligned with the transmitted beam or when there are radial misalignment errors.

The radius r_L represents the region where the irradiance becomes null, in the uniform beam model, or where it falls in the ratio $1/e^2$ of its peak, in the Gaussian beam model. This radius is normally called the spot size of the beam. In Fig. 2(b), that for the uniform beam model, the energy outside r_L will not be captured by the receiver.

2.1 *Simplified Uniform Model*

A simplified model for geometric attenuation consists of a circular energy beam with uniform distribution, instead of Gaussian. In this model, the geometric attenuation can be calculated by

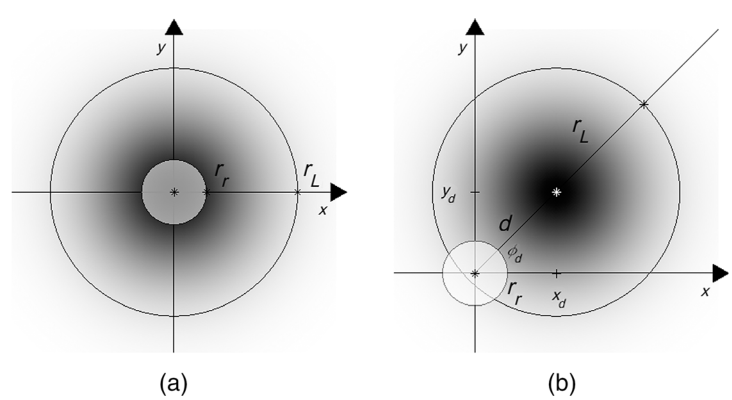

Fig. 2 Receiver and transmitted areas: (a) aligned and (b) misaligned.

the ratio between the area of the receiver and the area of the beam at the position of the receiver, $z = L$, considering the beam aligned with the z-axis. For a circular receiver, the expression for geometric attenuation (α_{geo}), which is widely used in link budgets, is as follows:[4]

$$\alpha_{\text{geo}} = \frac{r_r^2}{\left(r_t + \frac{\theta L}{2}\right)^2}, \tag{1}$$

where r_r is the receiver aperture radius, r_t is the transmitter aperture radius, θ is the beam divergence angle (mrad), and L is the range (km).

For the uniform beam model, a closed-form expression for attenuation by misalignment is already known.[12] This model, even inaccurate, is widely used in practice. However, in long-range links, when misalignment is observed, the uniform beam model leads to significant errors for the budget. Therefore, the accurate irradiance profile, with Gaussian energy distribution, is necessary.

2.2 *Gaussian Beam Model*

The wave solution for a Gaussian beam propagating along the z-axis, using cylindrical coordinate system, can be expressed as[3]

$$U(\rho, \phi, z) = A_0 \frac{r_t}{r_L(z)} \exp\left(-\frac{\rho^2}{r_L(z)^2}\right) \exp\left(-jkz - \frac{jk\rho^2}{2R(z)} + j\Phi\right), \tag{2}$$

where A_0 is a constant related to power at the transmitter output, $r_L(z)$ is the beam width along z, $R(z)$ is the curvature radius at a point z, and Φ represents a phase retardation relative to a uniform plane wave. Due to circular symmetry, there is no variation along the coordinate ϕ.

The optical intensity can be observed through the irradiance profile, given by $I(\rho, z) = |U(\rho, z)|^2$. Thus, from Eq. (2), we have

$$I(\rho, \phi, z) = A_0^2 \frac{r_t^2}{r_L^2(z)} \exp\left[-\frac{2\rho^2}{r_L^2(z)}\right], \tag{3}$$

where $r_L(z)$ can be approximated as $r_t + \frac{\theta z}{2}$.

Using Eq. (3) and neglecting absorption and scattering losses, the detected power is evaluated by integration of the irradiation profile over the surface of the receiver. The ratio between received power and total transmitted power, at the receiver plane, is usually known as geometric attenuation. When considering Gaussian beams, for a circular receiver with radius r_r, aligned with the transmitter, the geometric attenuation has a closed-form solution,[7]

$$\alpha_{\text{geo}} = \frac{\int_S I(\rho, \phi, z = L) \mathrm{d}S}{\int_0^\infty \int_0^{2\pi} I(\rho, \phi, z = L) \rho \mathrm{d}\phi \, \mathrm{d}\rho} = 1 - \exp\left[-\frac{2r_r^2}{r_L^2(L)}\right], \tag{4}$$

where S is the aperture area of the receiver.

The most natural way to include the effect of radial misalignment in the model is to translate axes over Eq. (3), written in rectangular coordinates. Thus, the peak of the translated Gaussian beam is positioned at the point (x_d, y_d), while the center of the receiver is at the origin of the coordinate system. Now, the power of the misaligned beam, detected at the receiver plane, can be written in rectangular coordinates, as follows:

$$P_{rx}(z = L) = \int_S A_0^2 \frac{r_t^2}{r_L^2} \exp\left[-\frac{2[(x - x_d)^2 + (y - y_d)^2]}{r_L^2}\right] \mathrm{d}S, \tag{5}$$

where x_d and y_d are the misalignments on the x- and y-axes, respectively.

Rewriting back Eq. (5) to cylindrical coordinates, making $x_d = d.\cos(\phi_d)$ and $y_d = d.\sin(\phi_d)$, where d is the absolute radial misalignment, and expanding the surface integral as a double integral over the circular receiver area, the power detected by the receiver is

$$P_{rx} = A_0^2 \frac{r_t^2}{r_L^2} \exp\left(\frac{-2d^2}{r_L^2}\right) \int_0^{r_r} \left\{\rho \cdot \exp\left(\frac{-2\rho^2}{r_L^2}\right) \int_0^{2\pi} \exp\left[\frac{4d.\rho.\cos(\phi - \phi_d)}{r_L^2}\right] \mathrm{d}\phi\right\} \mathrm{d}\rho. \tag{6}$$

Particularly, without loss of generality, since the area of integration is circular, we can make $\phi_d = 0$. Hence, the inner integral can be written as follows:

$$\int_0^{\pi} \exp\left[\frac{4d.\rho.\cos(\phi)}{r_L^2}\right]\mathrm{d}\phi + \int_{\pi}^{2\pi} \exp\left[\frac{4d.\rho.\cos(\phi)}{r_L^2}\right]\mathrm{d}\phi. \tag{7}$$

A substitution can be made in the last integral ($\phi' = \phi + \pi$). Indeed, both integrals resemble the modified Bessel function of the first kind and order $\alpha = 0$, in the integral form, given by

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} \exp[x.\cos(\phi)]\mathrm{d}\phi. \tag{8}$$

Thus, for $x = \frac{4d\rho}{r_L^2}$, as $I_0(x) = I_0(-x)$, Eq. (6) can be rewritten as

$$P_{rx} = A_0^2 \frac{r_t^2}{r_L^2} \exp\left(\frac{-2d^2}{r_L^2}\right)2\pi \int_0^{r_R} \left\{\rho \cdot \exp\left(\frac{-2\rho^2}{r_L^2}\right) \cdot I_0\left(\frac{4d\rho}{r_L^2}\right)\right\}\mathrm{d}\rho. \tag{9}$$

The integral in ρ from Eq. (9) is directly related to the Rice distribution. It is convenient to rewrite it as a difference of two improper integrals,

$$\int_0^{r_R} f(\rho)\mathrm{d}\rho = \int_0^{\infty} f(\rho)\mathrm{d}\rho - \int_{r_R}^{\infty} f(\rho)\mathrm{d}\rho. \tag{10}$$

As presented in Ref. 13,

$$\int_0^{\infty} x^{\alpha-1} \exp(-px^2)I_0(cx)\mathrm{d}x = \frac{1}{2}p^{\frac{-\alpha}{2}} \cdot \Gamma\left(\frac{\alpha}{2}\right) \cdot {}_1F_1\left(\frac{\alpha}{2}, 1, \frac{c^2}{4p}\right), \tag{11}$$

where ${}_1F_1$ is the confluent hypergeometric function and Γ is the gamma function.

When $x = \rho$, $\alpha = 2$, $p = \frac{2}{r_L^2}$, and $c = \frac{4d}{r_L^2}$:

$$\int_0^{\infty} \left[\rho \cdot \exp\left(\frac{-2\rho^2}{r_L^2}\right) \cdot I_0\left(\frac{4d\rho}{r_L^2}\right)\right]\mathrm{d}\rho = \frac{1}{2} \cdot \frac{r_L^2}{2} \cdot \Gamma(1) \cdot {}_1F_1\left(1,1,\frac{\frac{16d^2}{r_L^4}}{\frac{8}{r_L^2}}\right). \tag{12}$$

From the fact that ${}_1F_1(1,1,z) = \exp(z)$ and $\Gamma(1) = 1$, then Eq. (12) can be reduced to

$$\int_0^{\infty} \left[\rho \cdot \exp\left(\frac{-2\rho^2}{r_L^2}\right) \cdot I_0\left(\frac{4d\rho}{r_L^2}\right)\right]\mathrm{d}\rho = \frac{r_L^2}{4} \cdot \exp\left(\frac{2d^2}{r_L^2}\right). \tag{13}$$

From the work of Ref. 14, we know that

$$\int_b^{\infty} x^m \cdot \exp\left(-\frac{p^2x^2}{2}\right) \cdot I_{m-1}(ax)\mathrm{d}x = \frac{1}{a}\left(\frac{a}{p^2}\right)^{m-1} \cdot \exp\left(\frac{a^2}{2p^2}\right) \cdot Q_m\left(\frac{a}{p}, bp\right), \tag{14}$$

where Q_m is the Marcum Q-function of order m, defined as[14]

$$Q_m(a,b) = \int_b^{\infty} x\left(\frac{x}{a}\right)^{m-1} \cdot \exp\left(-\frac{x^2+a^2}{2}\right) \cdot I_{m-1}(ax)\mathrm{d}x, \quad m \geq 1. \tag{15}$$

When $m = 1$, $p = \frac{2}{r_L^2}$, $a = \frac{4d}{r_L^2}$, and $b = r_r$, Eq. (14) can be written as

$$\int_{r_r}^{\infty} \left[\rho \cdot \exp\left(\frac{-2\rho^2}{r_L^2}\right) \cdot I_0\left(\frac{4d\rho}{r_L^2}\right)\right]\mathrm{d}\rho = \frac{r_L^2}{4} \cdot \exp\left(\frac{2d^2}{r_L^2}\right) \cdot Q_1\left(\frac{2d}{r_L}, \frac{2r_r}{r_L}\right). \tag{16}$$

Now, using Eqs. (13)–(16), Eq. (9) can be rewritten as

$$P_{rx} = A_0^2 \frac{r_t^2}{r_L^2} \exp\left(\frac{-2d^2}{r_L^2}\right) \cdot 2\pi\left[\frac{r_L^2}{4} \exp\left(\frac{2d^2}{r_L^2}\right) - \frac{r_L^2}{4} \exp\left(\frac{2d^2}{r_L^2}\right) Q_1\left(\frac{2d}{r_L}, \frac{2r_r}{r_L}\right)\right], \tag{17}$$

which leads to

$$P_{rx} = A_0^2 \cdot \frac{\pi r_t^2}{2}\left[1 - Q_1\left(\frac{2d}{r_L}, \frac{2r_r}{r_L}\right)\right]. \tag{18}$$

When losses by absorption and atmospheric scattering are not considered, the geometric and misaligment attenuation are defined by $\alpha = P_{rx}/P_{tx}$, where P_{tx} is the total transmitted power of beam. The term outside the brackets in Eq. (18) is equal to the total transmitted power, as it was obtained in Eq. (13) by integrating the irradiance over an infinite plane $z = L$, in cylindrical coordinates. Consequently, the geometric attenuation and radial misalignment attenuation are

$$\alpha = \left[1 - Q_1\left(\frac{2d}{r_L}, \frac{2r_r}{r_L}\right)\right]. \tag{19}$$

Note that Eq. (19) has the same form of the Rician cumulative distribution function of r_r with shape parameter $K = 2d^2/r_L^2$ and scale parameter $\Omega = d^2 + r_L^2/2$.

2.3 *Other Considerations*

Using Eq. (15), for $d = 0$ and $m = 1$, Eq. (19) becomes equal to Eq. (4), as

$$Q_1\left(0, \frac{2r_r}{r_L}\right) = \int_{\frac{2r_r}{r_L}}^{\infty} x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x = \exp\left(-\frac{2r_r^2}{r_L^2}\right). \tag{20}$$

Hence, Eq. (19) is a general solution for Eq. (4) when considering misalignment. When only the misalignment attenuation is considered, its expression is as follows:

$$\alpha_{\mathrm{miss}} = \frac{1 - Q_1\left(\frac{2d}{r_L}, \frac{2r_r}{r_L}\right)}{1 - \exp\left(-\frac{2r_r^2}{r_L^2}\right)}. \tag{21}$$

The Marcum's Q-function is widely used to solve problems of radar detection, statistics, and probability theory. Although its formulation involves Bessel functions and improper integral, there are already consolidated algorithms capable of solving it with exceptional performance.[15,16] Thus, the direct use of Eq. (18) as a solution for Eq. (5) brings a performance gain in simulations of optical links with misalignment, especially in moving systems, where it is necessary to recalculate the integral in each iteration.

Equation (18) can also be used when there is a small angular misalignment, also known as tilt, between transmitter and receiver. To consider radial and angular misalignment in both directions (x and y), the received power can be calculated substituting d in Eq. (19) by

$$d = \sqrt{[x_c + L \cdot \tan(\gamma_y)]^2 + [y_c + L \cdot \tan(\gamma_x)]^2}, \tag{22}$$

where γ_y is the angle for a rotation on the y-axis and γ_x is the angle for a rotation on the x-axis.

3 Proof of Concept, Comparisons, and Discussion

To demonstrate the efficiency of the analytical solution in modeling the optical links with misalignment, two approaches have been proposed. First, as a proof of concept, an experiment was conducted to demonstrate that the attenuation due to misalignment is well modeled by the Gaussian beam model. The limitations of the simplified uniform model were also verified. In the sequence, three numerical simulations were carried out, to compare this analytical expression with other solutions, in terms of precision and execution time.

3.1 *Experimental Proof of Concept*

For the execution of this proof of concept, a laser source operating at 980 nm was positioned at the focal point of an OWC scope. This beam was transmitted over 2.82 m and captured by a similar scope, but with a phototransistor positioned at the focal point. Translational movements were performed, using a micrometer at the transmitter. Finally, standardized measurements were performed using an analog-to-digital converter on the voltage drop over a trimpot in series with a phototransistor emitter.

To characterize the source adequately, the beam pattern was measured at the receiver. The observed profile was circular and Gaussian, with a well-defined radius. The parameters used in this experiment are provided in Table 1.

Figure 3 shows the comparisons between the experimental results and the theoretical models. Figure 3(a) presents the attenuation curves as a function of radial misalignment for the experimental results, the theoretical model for Gaussian beam, and the simplified uniform model.

Table 1 Parameters of the experimental OWC system.

Parameter	Description	Value
r_r	Receiver radius	2.1 cm
r_L	Beam radius	1.65 cm
L	Link range	2.82 m

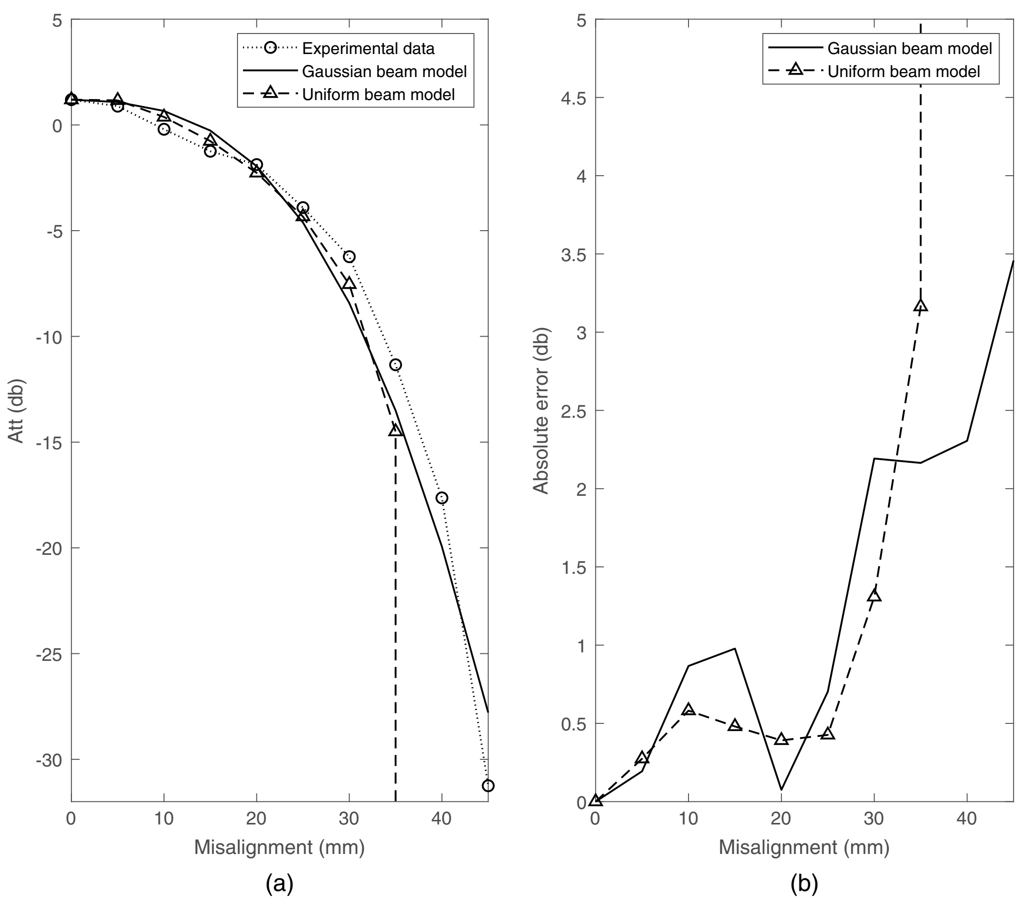

Fig. 3 Experimental and theoretical model for Gaussian beam misalignment. (a) Experimental and theoretical model for geometric and misalignment attenuation. (b) Absolute attenuation error.

It can be seen in Fig. 3(b) that, for misalignments smaller than the radius of the lens, both models present errors, in relation to the experiment, smaller than 1 dB. However, as the radial shift increases, causing a considerable decrease in the signal-to-noise ratio, the errors increase for both models. It should be noted that the uniform simplified model is not suitable for misalignment greater than $r_r + r_L$.

3.2 *Numerical Simulations*

As presented in Sec. 2, the proposed analytical solution involves calculating the Marcum Q-function. Currently, there are a few highly optimized algorithms capable of performing these calculations more efficiently than direct numerical integration.[16] In our work, the proposed analytical expression was calculated using the *marcumq* function, based on the algorithm presented in Ref. 15, present in MATLAB R2019b software. This algorithm has exceptional performance with quite precise results, as shall be noted later.

To demonstrate the precision of the closed-form expression presented in this paper, some comparisons were carried out. The analytical solution in Eq. (19) was compared with a purely numerical integration solution and with the power series solution, proposed in Ref. 9. The results for the uniform beam were also observed, as this model is still commonly used. Unless otherwise stated, all simulation results presented in this section considered the link parameters presented in Table 2, which are very close to commercial equipment currently available.

The numerical integration on Eq. (5) was performed through the *integral*2 function, which is based on the algorithm named TwoD, also present in MATLAB.[17] This function allows an absolute and relative error tolerance adjustment and was chosen as our reference, considering maximum error tolerance close to our computational limit, that is, both adjusted for the smallest positive normalized floating-point number (accessed by the command *realmin*). The power series solution was implemented directly and is subject to truncation errors according to the number of terms used.

In Fig. 4, the attenuation was calculated as a function of the radial misalignment between transceivers. Figure 4(a) shows our proposed solution and the other three previously mentioned solutions. The absolute errors between the reference and other Gaussian solutions are presented in Fig. 4(b).

In Fig. 4(a), it can be noticed that the simplified uniform model may not be accurate enough in some situations, in accordance with the results of Ref. 12, requiring the use of the Gaussian irradiance profile. It is also noticed that there is no visible error in calculating the Gaussian beam using numerical methods (reference), the analytical method, or by the power series solution (calculated for 300 terms). In fact, as shown in Fig. 4(b), the error is quite low, on the order of 10^{-10} dB for highly displaced links, which is comparable to the tolerance region for most numerical calculations.

As a second comparison, the attenuation was calculated for a case of 40 cm of misalignment between the transceivers, but for different values of receiver radius. Again, all four solutions are presented in Fig. 5, along with the absolute errors between our reference and the other two Gaussian solutions.

Table 2 Parameters of the simulated OWC system.

Parameter	Description	Value
r_t	Transmitter radius	2.5 cm
r_r	Receiver radius	10 cm
θ	Divergence angle	1 mrad
r_L	Beam radius	52.5 cm
L	Link range	1000 m

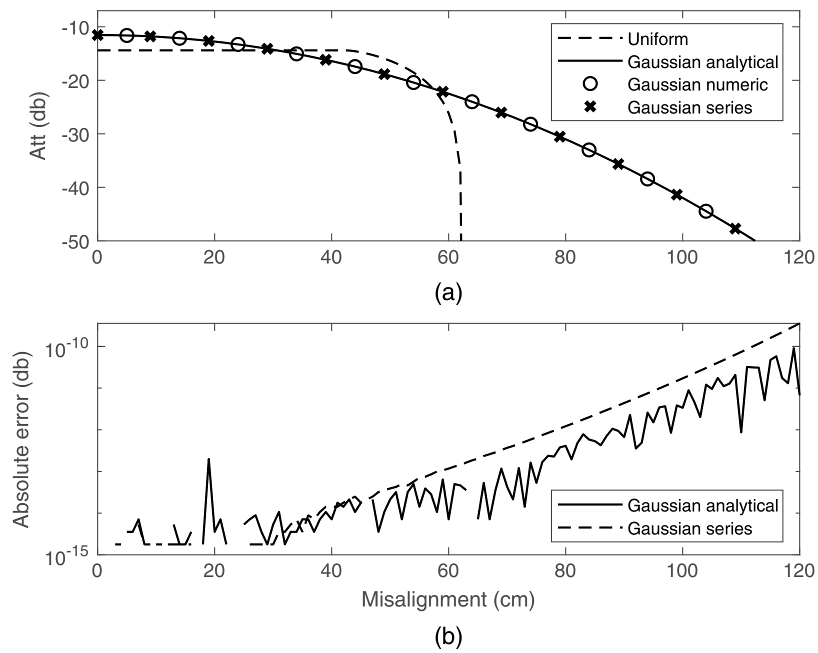

Fig. 4 (a) Attenuation by misalignment of tranceptors in dB. (b) Absolute attenuation error.

It is noticed in Fig. 5(a) that the uniform simplified model leads to errors up to 3 dB for this particular case. Again, Fig. 5(b) shows a good proximity between the numerical solution and the analytical or series solutions.

Finally, efficiency tests were carried to compare the presented methods. Thus, all solutions were simulated for a link with misalignment ranging from 0 to 1 m. The analyzed criteria were the average time of execution and the root-mean-square error (RMSE) in comparison to the reference. The execution time was measured using the *timeit* command, also present in MATLAB. This command measures the time required to run a function, but in a robust way, as it calls the function several times, returning the average between the measurements.

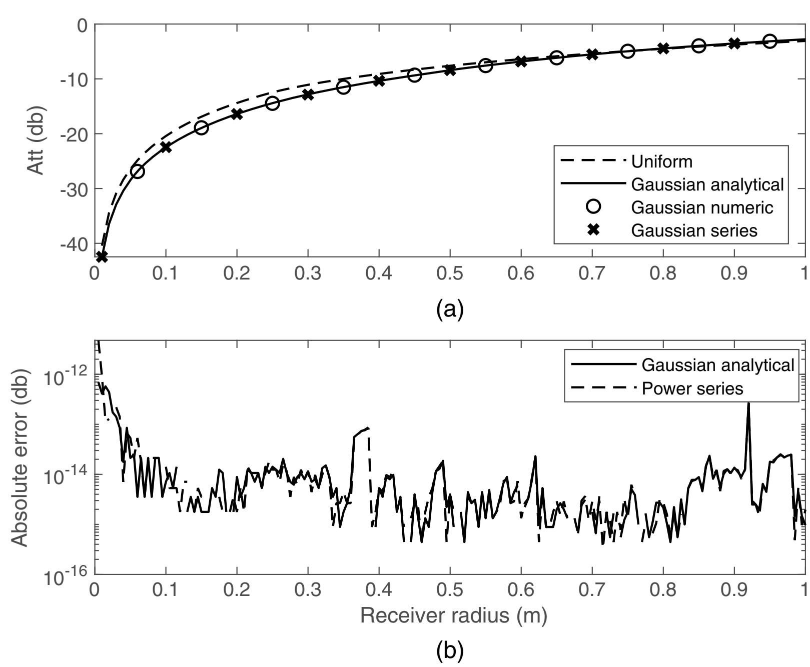

Fig. 5 (a) Attenuation for different receiver sizes. (b) Absolute attenuation error.

Table 3 Efficiency tests for each solution.

Calculation technique	RMSE (dB)	Average time (ms)
Numerical integration for maximum precision	0	1.93
Numerical integration for default precision	$5.35 \cdot 10^{-14}$	0.582
Series solution using 300 terms	$1.173 \cdot 10^{-16}$	116.42
Series solution using 7 terms	$1.501 \cdot 10^{-16}$	0.0791
Series solution using 1 term	$9.002 \cdot 10^{-6}$	0.00903
Analytical solution: Marcum Q-function	$4.26 \cdot 10^{-17}$	0.00733

All simulations were performed in MATLAB R2019b software, using Windows 10 as the operating system, on a machine with an Intel Core i7-7700HQ CPU and 16 GB RAM.

Table 3 presents the main results of these simulations. It is possible to check the RMSE and the average execution time for each technique. The first technique is the reference itself, which has a relatively high execution time for maximum precision. The second is the same numerical solution, but using the default setting for the *integral*2 function in MATLAB, which uses 10^{-10} and 10^{-6} as absolute and relative error tolerances, respectively. The series solution was truncated at different terms, to show its compromise between precision and time. The last line of the table has our proposed analytical solution, calculated using the *marcumq* function.

With these results, it is possible to observe the improvement in performance brought by the analytical solution for problems of misalignment of Gaussian beams. The series solution took a long time and did not achieve the same precision, even using 300 terms. The numerical integration solution, used as a reference, would also present numeric errors when modeling the real phenomenon, and it is about 263 times slower than the proposed solution. On the other hand, when trying to achieve the same computational time, as the implemented Marcum Q-function, the RMSE becomes too high.

4 Summary and Conclusions

In this work, we present the mathematical deduction of an analytical solution to the problem of calculating the attenuation by misalignment of Gaussian beams, which has a direct application in OWC systems. Although the presented solution involves the calculation of the Marcum Q-function, we demonstrated through simulations that the current algorithm allows a considerable performance gain in relation to other solutions, without losing precision. It is also demonstrated, through a small experiment, that the analytical solution presented is quite satisfactory for the modeling of an OWC system.

In conclusion, this solution proved to be more efficient than the calculation methods used so far, offering a more accurate and fast implementation option than others. As said before, the Marcum Q-function is widely used in problems of radar detection, statistics, probability theory, or whenever a shifted circular Gaussian function must be integrated. Given its simplicity and generality, we believe it should also become a reference solution in the field of optics.

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